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Finding a central vertex in an HHD-free graph ☆

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Abstract

In a graph $G=(V,E)$, the eccentricity $e(v)$ of a vertex v is $\max\{d(v,u)\colon u\in V\}$. The center of a graph is the set of vertices with minimum eccentricity. A house–hole–domino-free (HHD-free) graph is a graph which does not contain the house, the domino, and holes (cycles of length at least five) as induced subgraphs. We present an algorithm which finds a central vertex of a HHD-free graph in $O(\Delta^{1.376}|V|)$ time, where Δ is the maximum degree of a vertex of G. Its complexity is linear in the case of weak bipolarizable graphs, chordal graphs, and distance-hereditary graphs. The algorithm uses special metric and convexity properties of HHD-free graphs.

Keywords: Distances in graphs; Central vertex; Efficient algorithms; Chordal graphs; HHD-free graphs

1. Introduction

The problem we address in this paper, to find a vertex in a given graph G whose maximum distance to any vertex of G is minimized (a *central vertex* of G), is one of the basic facility location problems. As yet, no efficient algorithm for this problem in general graphs, avoiding the computation of the whole distance matrix, has been designed. Linear time algorithms for finding a central vertex have been presented for trees [11,12,21], 2-trees and maximal outerplanar graphs [10,20], strongly chordal graphs [6], interval graphs [19], chordal graphs [4], dually chordal graphs [7], distance hereditary graphs [8], and claw-free asteroidal triple-free graphs [13].

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In this paper, we present an $O(\Delta^{1.376}|V|)$ time algorithm that finds a central vertex of a *house–hole–domino–free* (*HHD-free*) *graph* $G=(V,E)$, where Δ is the maximum degree of a vertex of G. The algorithm works in linear time for weak bipolarizable graphs and for distance-hereditary graphs—two well-known subclasses of HHD-free graphs. HHD-free graphs were first introduced and investigated by Jamison and Olariu [16] (see also [14]). HHD-free graphs represent a common generalization of all aforementioned classes of graphs, except dually chordal graphs and claw-free asteroidal triple-free graphs. The key idea of our algorithm is similar to that we applied in the case of chordal graphs [4]: given a HHD-free graph G, with a few applications of breadth-first-search we can find two mutually farthest vertices x and y, such that the distance $d(x,y)$ is at most 3 less than the diameter of G. Intuitively, the set of all middle vertices of shortest x,y-paths represents a "small" region of G where some central vertices can be located. Selecting in some sense a "best" vertex c of this region, we prove that either c indeed is central or the eccentricity of any vertices farthest from c is larger than $d(x,y)$. In the latter case, we improve the value $d(x,y)$ and continue our search with a new pair of mutually farthest vertices. After at most three improvements, we will come to a central vertex of G. The correctness proof of the algorithm requires some additional properties of HHD-free graphs which we present in the next section.

2. Preliminaries

All graphs occurring in this note are connected and simple, i.e., finite, undirected, loopless and without multiple edges. In a graph $G=(V,E)$ the *length* of a path from a vertex v to a vertex u is the number of edges in the path. The *distance* $d(u,v)$ between u and v is the length of a minimum length path from u to v and the *interval* $I(u,v)$ between u and v is the set of all vertices lying on shortest paths connecting u and v, i.e.,

$$I(u,v)=\{w\in V:\ d(u,v)=d(u,w)+d(w,v)\}.$$

The *eccentricity* $e(v)$ of a vertex v is the maximum distance from v to any vertex in G. Denote by $F(v)$ the set of all vertices farthest from v, i.e.,

$$F(v)=\{w\in V:\ d(v,w)=e(v)\}.$$

We will say that the vertices x and y are *mutually farthest* if $e(x)=d(x,y)=e(y)$. The *radius* $r(G)$ is the minimum eccentricity of a vertex in G and the *diameter* $d(G)$ is the maximum eccentricity. The *center* $C(G)$ of G is the subgraph induced by the set of all *central vertices*, i.e., vertices whose eccentricities are equal to $r(G)$. The *disk* of radius k centered at v is the set of all vertices at distance at most k to v:

$$D(v,k)=\{w\in V:\ d(v,w)\leqslant k\}.$$

Obviously, $C(G)=\bigcap_{v\in V} D(v,r(G))$ for any graph G.

A graph G is *house–hole–domino-free* (*HHD-free*) if it does not contain the house, the domino, and holes (cycles of length at least 5) as induced subgraphs. A HHD-free graph which does not contain the "*A*" of Fig. 1 as an induced subgraph is called *weak*

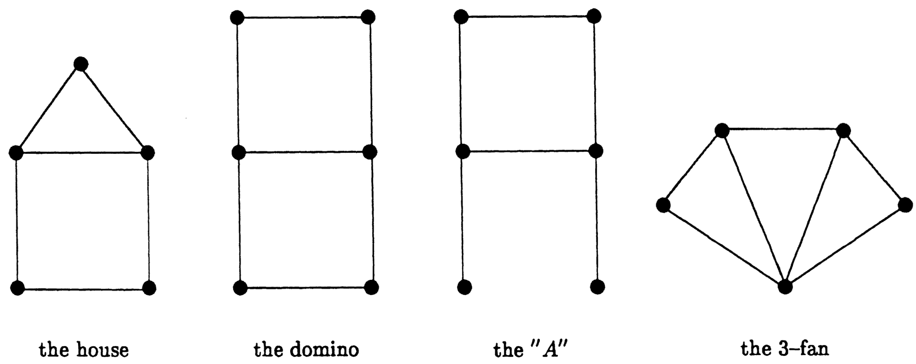

Fig. 1. Special graphs.

bipolarizable [18]. A *distance-hereditary graph* is a HHD-free graph that does not contain the 3-fan as an induced subgraph [15]. Recall also that a graph is *chordal* if every induced cycle is of length 3.

A subset $S \subseteq V$ of a graph $G=(V,E)$ is called m^3-*convex* [9] if for any pair of vertices $u,v \in S$ each induced path of length at least 3 connecting u and v is contained in S. Notice that m^3-convex sets are not necessarily connected and that the family of m^3-convex sets is closed under taking intersections.

Lemma 1 (Dragan et al. [9]). *Any disk of a HHD-free graph G is* m^3-*convex.*

A graph G is *weakly modular* [1,3] if its metric d satisfies the following two conditions:

Triangle condition: For any three vertices u,v,w with

$$1=d(v,w)<d(u,v)=d(u,w)=k$$

there exists a common neighbour x of v and w such that $d(u,x)=d(u,v)-1=k-1$

Quadrangle condition: For any four vertices u,v,w,z with

$$d(v,z)=d(w,z)=1, d(v,w)=2 \quad \text{and} \quad k=d(u,v)=d(u,w)=d(u,z)-1,$$

there exists a common neighbour x of v and w such that $d(u,x)=d(u,v)-1=k-1$.

Lemma 2. *Any HHD-free graph G is weakly modular.*

Proof. The result follows from [3]. To make the presentation self-contained we give a direct proof. First we verify the quadrangle condition. Pick neighbours v' of v and w' of w on shortest paths between v,u and w,u, respectively. We can suppose that $v' \neq w'$, $v'w \notin E$ and $w'v \notin E$, otherwise there is nothing to prove. Since $v',w' \in D(u,k-1)$ and $D(u,k-1)$ is m^3-convex, the path $v'vzww'$ cannot be induced. From the choice of v and w we conclude that v' and w' must be adjacent, thus yielding an induced 5-cycle.

Now to the triangle condition: let v' and w' be the neighbours of v and w, respectively, in the disk $D(u,k-1)$. Again, we can suppose that $v' \neq w'$, $v'w \notin E$ and $w'v \notin E$. By m^3-convexity of $D(u,k-1)$ we deduce that the vertices v' and w' must be adjacent. Since $d(v',u)=d(w',u)=k-1$, we may assume, by induction, that there is a common neighbour u' of v' and w' at distance $k-2$ to u. But then u',v',w',v,w induce a house. □

Lemma 3. *Let u,v and w be vertices of a HHD-free graph G such that $d(v,w)=2$ and $d(u,w)=d(u,v)+1$. Then, there exist two vertices x and y such that v, x and y are pairwise adjacent, $d(y,u)=d(u,v)-1=d(x,u)-1$, and x is adjacent to w.*

Proof. Let $k=d(u,v)$. First, we will show that $D(u,k)$ contains a common neighbour of v and w. Assume the contrary, and let z be a neighbour of w in $D(u,k)$. Pick a common neighbour p of v and w. Since $p,w \notin D(u,k)$, the path $vpwz$ cannot be induced. Our assumption implies that only the vertices z and p of this path can be adjacent. Then, $v,z \in I(p,u)$ and, by the quadrangle condition, there exists a common neighbour q of z and v at distance $k-1$ to u. Since w,p,z,v,q do not induce a house, q and w must be adjacent, contrary to the assumption. So, let x be a common neighbour of v and w in $D(u,k)$. Since $d(v,u)=d(x,u)$, by the triangle condition, there exists a vertex y which is adjacent to both v and x and has distance $k-1$ to u. □

For a subset $S \subset V$ and a vertex $v \notin S$, we denote by

$$Pr(v,S)=\{y \in S\colon\ d(v,y)=d(v,S)\}$$

the *metric projection* of v on S (recall that $d(v,S)=\min\{d(v,w)\colon\ w \in S\}$).

Lemma 4. *Let S be a m^3-convex set of a HHD-free graph G. For any vertex $v \notin S$, there exists a vertex v^* at distance $d(v,S)-1$ to v which is adjacent to all vertices of $Pr(v,S)$.*

Proof. Put $k=d(v,S)$ and let v^* be a vertex at distance $k-1$ to v which is adjacent to maximum number of vertices of $Pr(v,S)$. Suppose that v^* is not adjacent to some vertex $u \in Pr(v,S)$. Pick a vertex $x \in Pr(v^*,S)$ and a neighbour y of u in the disk $D(v,k-1)$. We assert that y and x are adjacent. Suppose the contrary. If x and u were nonadjacent, then consider an induced path between x and u passing through v^*, some vertices of $I(v^*,v)\cup I(y,v)$, and y. Its length is at least three, contrary to m^3-convexity of S. Therefore, x and u are adjacent. From m^3-convexity of $D(v,k-1)$ we deduce that the vertices v^* and y are adjacent, too. By the triangle condition, there is a common neighbour of v^* and y at distance $k-2$ to v. This vertex together with v^*,y,x and u induces a house. Hence, y must be adjacent to any neighbour x of v^* in $Pr(v,S)$. Since, in addition, y is adjacent to u, this contradicts the choice of v^*. □

Following [2], a graph G satisfies the metric condition (α_i) if for any four vertices u,v,w,x such that $v \in I(u,w), w \in I(v,x)$ and $d(v,w)=1$ we have

$$d(u,x) \geqslant d(u,v)+d(w,x)+1-i.$$

Lemma 5. *Any HHD-free graph G satisfies the condition* (α_2). *Moreover*

(1) $d(u,x) = d(u,v) + d(w,x) - 1$ *holds iff any neighbour v' of v in $I(v,u)$ and any neighbour w' of w in $I(w,x)$ are adjacent*;
(2) $d(u,x) = d(u,v) + d(w,x)$ *holds iff there exist vertices $y \in I(v,u)$ and $z \in I(w,x)$ such that y,z lie on a shortest path between u and x and the vertices v,w,y,z and some other vertex t induce a 3-fan* (*Fig.* 1).

Proof. Let $d(v,u)=k$, $d(w,x)=l$, and suppose that $d(u,x) \leqslant k+l$. First, we show that $d(u,x) \geqslant k$. Indeed, otherwise $v,x \in D(u,k)$, while $w \notin D(u,k)$. Since $w \in I(v,x)$, m^3–convexity of $D(u,k)$ implies $l=1$. Then $d(w,u) \leqslant k$, contrary to $v \in I(u,w)$. Consider a vertex $t \in I(u,x)$ at distance k to u. First suppose that t and v are not adjacent. Then, any induced path which connects t and v and passes through w has length 2. Since $d(t,x) \leqslant l$, the vertex v cannot be adjacent to a vertex of $I(t,x) \cup I(x,w) \setminus \{w\}$. Therefore, vwt is the unique induced path connecting v and t and going through w. In this case, $d(t,x) \geqslant l-1$ and $d(u,x) \geqslant k+l-1$. So, it remains to verify the assertions (1) or (2). Since $v,t \in I(w,u)$, by the quadrangle condition, there is a common neighbour p of v and t at distance $k-1$ to u. We distinguish between two cases depending on the value of $d(t,x)$.

Case 1: $d(t,x) = l-1$, *i.e.*, $d(u,x) = k+l-1$. Pick two vertices $v' \in I(v,u)$ and $w' \in I(w,x)$ adjacent to v and w, respectively. As $v', p \in I(v,u)$ and $t, w' \in I(w,x)$, by weak modularity of G we can find the vertices $u' \in I(v',u) \cap I(p,u)$ and $x' \in I(w',x) \cap I(t,x)$ which are adjacent to v', p and w', t, respectively. From the initial distance requirements and since G is HHD-free we deduce that v' is adjacent to t and w' is adjacent to p. Applying this argument again, we obtain that the vertices v' and w' must be adjacent as well, thus establishing (1).

Case 2: $d(t,x) = l$, *i.e.*, $d(u,x) = k+l$. Since t and w are equidistant to x, by the triangle condition, we can find a common neighbour x' of t and w at distance $l-1$ to x. The vertices x' and p cannot be adjacent, for otherwise

$$d(u,x) \leqslant d(x,x') + 1 + d(p,u) = l - 1 + 1 + k - 1 = k + l - 1,$$

which is impossible. But then v, w, p, t and x' induce a house, and thus case 2 is impossible.

Finally, suppose that v and t are adjacent. Since $d(v,x) = l+1$ and $d(u,x) \leqslant k+l$, necessarily $d(x,t)=l$ and $d(u,x)=k+l$. We continue by verifying (2). Applying weak modularity of G to v,t,u and to $w,t \in I(v,x)$, we can find a vertex $y \in I(v,u) \cap I(t,u)$ adjacent to v,t and a vertex $z \in I(w,x) \cap I(t,x)$ adjacent to w,t. Since G is house-free and $d(u,x)=k+l$, we deduce that w and t are adjacent, whence the vertices y,v,w,z,t induce a 3-fan. This finishes the proof. □

Lemma 6. *For any vertex v of a HHD-free graph G and any farthest vertex $u \in F(v)$, we have $e(u) \geqslant 2r(G) - 3$.*

Proof. Assume the contrary and among the vertices which fail the assertion choose a vertex v with minimal eccentricity. Then $e(u) < 2r(G) - 3$ for a vertex $u \in F(v)$.

Let $d(v,u) = k$ and denote $X = I(v,u) \cap D(v,1)$. From our assumption, we deduce that if $x \in X$ then $u \notin F(x)$, i.e., $e(x) \geqslant e(v)$. If for some $x \in X$ we can find a vertex $z \in F(x) \setminus F(v)$, then $v \in I(z,x)$ and by Lemma 5,

$$d(u,z) \geqslant e(v) - 1 + e(x) - 2 \geqslant 2e(v) - 3 \geqslant 2r(G) - 3.$$

Hence, $e(x) = e(v)$, and $F(x) \subseteq F(v)$ for all vertices $x \in X$. Let x^* be a vertex of X having the minimum number of farthest vertices and let $y^* \in F(x^*)$. Since $d(v,y^*) = d(x^*,y^*)$, by the triangle condition, there is a vertex $x^+ \in I(v,y^*) \cap I(x^*,y^*)$ adjacent to both v and x^*. If $x^+ \notin I(v,u)$, then $x^* \in I(x^+,u)$ and, by the condition (α_2), $d(u,y^*) \geqslant 2e(v) - 3 \geqslant 2r(G) - 3$. Hence, we may assume that $x^+ \in I(v,u)$. From our choice of x^* there exists a vertex $y^+ \in F(x^+) \setminus F(x^*)$. Since x^+ and x^* are adjacent and equidistant to u, by the triangle condition we get a vertex $w \in I(x^+,u) \cap I(x^*,u)$ adjacent to x^+ and x^*. Since $x^* \in I(x^+,y^+)$ and $x^+ \in I(x^*,y^*)$, we conclude that $d(w,y^*) \geqslant e(v) - 1$ and $d(w,y^+) \geqslant e(v) - 1$.

If $x^+ \in I(w,y^*)$ or $x^* \in I(w,y^+)$, by the condition (α_2), at least one of the following inequalities holds:

$$d(u,y^*) \geqslant d(x^+,y^*) + d(w,u) - 1 \geqslant 2e(v) - 4 \geqslant 2r(G) - 4,$$

$$d(u,y^+) \geqslant d(x^*,y^+) + d(w,u) - 1 \geqslant 2e(v) - 4 \geqslant 2r(G) - 4.$$

According to Lemma 5 $d(u,y^*) = 2r(G) - 4$ holds only if any neighbours $x' \in I(x^+,y^*)$ and $w' \in I(w,u)$ of x^+ and w, respectively, are adjacent. But in this case the vertices x^*, x^+, w, x', w' induce a house. So, we can suppose that $d(w,y^*) = d(x^+,y^*)$ and $d(w,y^+) = d(x^*,y^+)$. Again, by the triangle condition, there exist the vertices $s \in I(x^+,y^*) \cap I(w,y^*)$ and $t \in I(x^*,y^+) \cap I(w,y^+)$, which are adjacent to x^+, w and x^*, w, correspondingly. If $s,t \in D(u,k-2)$, then the path sx^+x^*t cannot be induced. Thus, the vertices s and t must be adjacent. Now, we have constructed a house induced by s, x^+, x^*, t, v. So, without loss of generality, let $w \in I(t,u)$. By (α_2), $d(u,y^+) \geqslant d(u,w) + d(t,y^+) \geqslant 2r(G) - 5$. According to Lemma 5 $d(u,y^+) = 2r(G) - 5$ holds only if any neighbours $w' \in I(w,u)$ of w and $t' \in I(t,y^+)$ of t would be adjacent. Again, we get an induced house. Thus, $d(u,y^+) = d(u,w) + d(t,y^+) = 2r(G) - 4$. Then there is a vertex p which together with w,t and some vertices $w' \in I(w,u)$ and $t' \in I(t,y^+)$ induces a 3-fan. Notice that p and s are nonadjacent, otherwise in the subgraph induced by v, x^+, x^*, s, p and t we can find either an induced 5-cycle or a house. We distinguish between two cases.

Case 1: $d(s,u) = d(w,u)$. Since $s,p \in D(u,k-2)$ and $D(u,k-2)$ is m^3-convex, the path sx^+x^*tp must contain at least two chords. As $x^+ \in I(x^*,y^*)$ and $x^* \in I(x^+,y^+)$, only px^* can be a chord, a contradiction.

Case 2: $d(s,u) = d(w,u) + 1$. Applying the same arguments to the vertices y^*, s, w, u (as before to y^+, t, w, u), we can find a vertex q adjacent to s, w and to some neighbours of s and w in the intervals $I(s,y^*)$ and $I(w,u)$. The vertices q and t are not adjacent, otherwise in the subgraph induced by q, s, x^+, x^*, t, and v we can find a forbidden house or 5-cycle. Since $p,q \in D(u,k-2)$ and $s, x^+, x^*, t \notin D(u,k-2)$, by m^3-convexity of $D(u,k-2)$, the path qsx^+x^*tp is not induced. It is easy to see that only px^*, qx^+ and pq are potential chords of this path. Then, we get an induced 5-cycle or, if all three chords assist, the vertices v, x^+, x^*, q, p induce a house, and final contradiction arises. □

3. Finding a central vertex

In this section, we present the contribution of this paper. First, we outline the entire algorithm to compute a central vertex of a HHD-free graph $G=(V,E)$. The correctness and the details of its implementation are subsequently discussed.

3.1. The algorithm

Algorithm. Finding a central vertex of a HHD-free graph

Input: A HHD-free graph $G=(V,E)$

Output: A central vertex c of G

1. Find a pair of mutually farthest vertices x, y and let $delta = d(x, y)$.
2. Construct the set $S = D(x, \lfloor delta/2 \rfloor) \cap D(y, delta - \lfloor delta/2 \rfloor)$.
3. Compute the value $R = \max\{d(v,S)\colon v \in V\}$.
4. Find the sets $F' = \{v \in V\colon d(v,S) = R\}$ and $F'' = \{v \in V\colon d(v,S) = R-1\}$.
5. Determine the set S^* of all vertices of S which belong to maximum number of metric projections of vertices from F'.
6. Among the vertices of S^* find a vertex c for which the set $\{u \in F''\colon d(c, Pr(u,S)) \leqslant 1\}$ has maximum cardinality.
7. Let u be an arbitrary vertex from $F(c)$. If $e(u) > delta$, then replace the pair x, y by a new pair of mutually farthest vertices at distance larger than $delta$ and repeat steps 1–7, otherwise c is a central vertex of G.

Next, we will discuss the details of the algorithm. We start with the computation of a pair of mutually farthest vertices. To do this, we pick an arbitrary vertex v of G and find a vertex $x \in F(v)$. Such a vertex x can be selected by breadth-first-search (BFS) which starts from the vertex v. Now, starting the BFS from x we will find a vertex $y \in F(x)$. From Lemma 6, we know that $2r(G)-3 \leqslant d(x,y) \leqslant d(G) \leqslant 2r(G)$. If x and y are mutually farthest, then we go to the next step. Otherwise, $d(x,y) < e(y)$, and we repeat the above procedure for $v := y$. In at most two repetitions we will arrive at a pair x, y of mutually furthest vertices. Let $delta = d(x,y)$. The value $\lfloor (delta+1)/2 \rfloor$ already represents a sharp approximation of the radius of G: it equals $r(G)$ or $r(G)-1$. Therefore, we can find a pair x, y of mutually farthest vertices and an approximation of the radius of G in linear time $O(|V| + |E|)$. For given x and y in step 2, we construct the set S of vertices which are suspected to be central at this iteration of the algorithm. Namely, S consists of all vertices $w \in V$ such that $d(x,w) = \lfloor delta/2 \rfloor$ and $d(y,w) = delta - \lfloor delta/2 \rfloor$.

To implement steps 3–6 first for each vertex $v \in V$ we compute the following three parameters: the distance $dist(v)$ from v to S, the cardinality $num(v)$ of the metric projection $Pr(v,S)$, and, finally, in $gate(v)$ we keep a vertex adjacent to all vertices of $Pr(v,S)$ and having distance $dist(v)-1$ to v. Since S is m^3-convex as an intersection of two disks (see step 2), by Lemma 4, a vertex with this property necessarily exists and we call it a *gate* of v in the set S. Let $N^i(S) = \{v \in V\colon\ d(v,S) = i\}$. Since $N^{i+1}(S)=N^1(N^i(S))$ these sets can be computed in $O(|V|+|E|)$ total time. In particular, we can find the value R and the sets F' and F'' within these time bounds. For any vertex $v \in N^1(S)$, we put $dist(v) = 1$, $gate(v) = v$ and $num(v) = |D(v,1) \cap S|$. Let us suppose that we have computed these parameters for all vertices from the set $N^{i-1}(S)$. Then, for each vertex $v \in N^i(S)$, we have

$$dist(v) = i, \quad gate(v) = gate(u), \quad num(v) = num(u),$$

where u is a neighbour of v in $N^{i-1}(S)$ with the maximum parameter $num(u)$. Correctness of this procedure follows from the next property of HHD-free graphs.

Lemma 7. *Let S be a m^3-convex set of a HHD-free graph G and let v be a vertex of G with $d(v,S) = k \geqslant 2$. Then, for any two neighbours $p,q \in N^{k-1}(S)$ of v, the metric projections $Pr(p,S)$ and $Pr(q,S)$ are comparable by inclusion.*

Proof. Suppose not: then there exist the vertices $a \in Pr(p,S)\backslash Pr(q,S)$ and $b \in Pr(q,S)\backslash Pr(p,S)$. Consider a path which consists of a shortest path between a and p, the edges pv and vq, and a shortest path between q and b. Evidently, this path intersects S only in the vertices a and b. From the choice of a and b and m^3-convexity of S we deduce that a and b are adjacent. The m^3-convexity of $D(v,k-1)$ implies that the neighbours $a',b' \in D(v,k-1)$ of a and b must be adjacent, too. Since $d(a',v) = d(b',v)$, by the triangle condition we can find a common neighbour t of a' and b' at distance $k-2$ to v. As a result, we get a house induced by the vertices a,b,a',b',t. □

With the parameters $gate(v)$ and $num(v)$ in hands, we can efficiently implement step 5. To find the set S^*, for each vertex $v \in N^1(S)$, we compute the number $n'(v)$ of vertices $u \in F'$ such that $gate(u) = v$. Now, for any vertex $s \in S$ we count the sum of values $n'(v)$ over $v \in D(s,1) \cap N^1(S)$. Then S^* consists of those vertices of S for which this sum is maximal. The complexity of this procedure is $O(|V| + |E|)$. Step 6 is harder. Unfortunately, for all HHD-free graphs, we were not able to implement it in linear time. A straightforward approach is to find first for each vertex of S^* all vertices at distance at most 2 from it, and then to use again the arguments above. Namely, for each vertex $v \in N^1(S)$, compute the number $n''(v)$ of vertices $u \in F''$ with $gate(u) = v$; for any vertex $s \in S^*$ count the sum of values $n''(v)$ over $v \in D(s,2) \cap N^1(S)$; and choose a vertex $c \in S^*$ for which this sum is maximal. Correctness of this procedure follows from the fact that in HHD-free graphs $d(s,Pr(u,S)) \leqslant 1$ holds if and only if $d(s,gate(u)) \leqslant 2$ (see Lemma 11). Since the computation of $D(s,2)$ for all $s \in S^*$ can be done totally in $O(|V|^\alpha)$ time using Boolean matrix multiplication, the complexity of step 6 is at most $O(|V|^\alpha)$. Currently, $\alpha = 2.376$ [5]. Using an idea of Kloks [17] one can implement step 6 in $O(\Delta^{\alpha-1}|V|)$ time, where Δ is the maximum degree of a vertex of G. For this, let $U = \{u \in V\colon\ u = gate(v) \text{ for some } v \in F''\}$, take $k = (\alpha-1)|U|/|S|$, and proceed as follows:

1. Partition U into k sets $U_1,\dots,U_k$ of approximately equal sizes.

2. Using the matrix multiplication (on U_i with S), for each U_i and $s \in S^*$ determine the number $p_i(s)$ of vertices from F'' having gates in U_i and for which $d(s, Pr(u,S)) \leqslant 1$.
3. For each $s \in S^*$, compute $p(s) = p_1(s) + \cdots + p_k(s)$.
4. Choose as c a vertex of S^* with maximal $p(c)$.

Clearly, step 2 takes most of the time. Its complexity is $O(|S|^{\alpha-1}|U|)$, because there are k matrix multiplications with each matrix of size $|S| + |U|/k$. Since $|U| \leqslant |V|$ and $|S| \leqslant \Delta$ (by Lemma 10), we conclude that the number of operations necessary to find the vertex c is $O(\Delta^{\alpha-1}|V|)$. Later, we will show that, for weak bipolarizable graphs (and hence, for chordal graphs) step 6 of the algorithm is superfluous (it is enough to take as c an arbitrary vertex of S^*), and for distance-hereditary graphs step 6 can be implemented in linear time.

Since initially $delta \geqslant 2r(G) - 3$, there are at most three returns from step 7 to step 1. Hence, the algorithm requires in total $O(\Delta^{\alpha-1}|V|)$ time. Note that all steps of the algorithm except step 6 have linear time bounds. In order to prove the correctness of the algorithm it suffices to show that if $e(c) > \lfloor delta/2 \rfloor + 1$, then $e(z) > delta$ for any vertex $z \in F(c)$, otherwise c is a central vertex of G.

3.2. Correctness of the algorithm

We now come to proving the correctness of our algorithm. We will assume that mutually furthest vertices x and y are at distance greater than 2. If $delta = d(x,y) \leqslant 2$ then either vertices x and y are central or there exists a vertex $z \in I(x,y)$ which is adjacent to all vertices of G. In the second case z is a central vertex of G.

Lemma 8. *If delta is even* ($delta = 2k$), *then* $k \leqslant R \leqslant k+1$. *Moreover, if* $k = r(G)$ *then* $R = k$.

Proof. Suppose by way of contradiction that $d(u,S) = R \geqslant k+2$ for some vertex u. Pick a vertex $v \in Pr(u,S)$ and some of its neighbours w in the interval $I(u,v)$. Then either $w \in I(v,x) \cup I(v,y)$ or $w \notin I(x,y)$. If say $w \in I(v,x)$, then $v \in I(w,y)$ and, by (α_2),

$$d(y,u) \geqslant d(y,v) + d(w,u) - 1 \geqslant 2k.$$

Since $d(x,y) \geqslant d(y,u)$, we conclude that $d(y,u) = 2k$. The equality holds only if $R = k+2$ and any neighbours $w' \in I(w,u)$ and $v' \in I(v,y)$ of w and v, respectively, are adjacent. Hence, $w' \in S$, contrary to the choice of v from $Pr(u,S)$.

Now let $w \notin I(x,y)$. Then $v \in I(w,x) \cup I(w,y)$, say $v \in I(w,y)$. By condition (α_2),

$$d(y,u) \geqslant d(y,v) + d(w,u) - 1 = k + R - 2 \geqslant 2k.$$

Again, since $2k = d(x,y) \geqslant d(y,u)$, we obtain $d(y,u) = 2k$. Therefore, $d(w',v') = 1$ for any vertices $w' \in I(w,u)$ and $v' \in I(v,x)$ adjacent to w and v, respectively. Since $w \notin I(x,y)$, necessarily $d(x,w) \geqslant k$. If $d(w,x) = k$, by the triangle condition, there is a common neighbour t of w and v at distance $k-1$ to x. Then t must be adjacent to w', otherwise we get a house induced by w', v', w, v and t. But then again $w' \in S$, contrary to the choice of v. So, $d(w,x) = k+1$. By Lemma 5, the vertex w' must be adjacent to any neighbour $v'' \in I(v,x)$ of v, otherwise

$$d(x,u) > d(x,v) + d(v,u) - 2 \geqslant 2k.$$

which is impossible. If w' and v'' are adjacent, then v', w' and v'' lie on a common shortest path between x and y. Then $w' \in S$, contrary to the choice of v.

Finally, assume that $delta = 2k = 2r(G) = d(x,y)$. Since $C(G) \subseteq S = D(x, r(G)) \cap D(y, r(G))$, we conclude that $R = r(G) = k$. □

Lemma 9. *If $delta$ is odd ($delta = 2k-1$), then $R = k$.*

Proof. Suppose by way of contradiction that there exists a vertex u for which $d(u,S) \geqslant k+1$ and let $v \in Pr(u,S)$. Pick a neighbour w of v in the interval $I(v,u)$. We distinguish between two cases.

Case 1: $w \in I(x,y)$. Then $w \in I(x,v) \cup I(v,y)$. If $w \in I(x,v)$, then $v \in I(w,y)$, and, by (α_2),

$$d(y,u) \geqslant d(y,v) + d(v,u) - 2 \geqslant 2k-1.$$

Since $d(y,u) \leqslant delta$, necessarily $R = k+1$ and $d(y,u) = 2k-1$. By Lemma 5, this implies that any $w' \in I(w,u)$ and $v' \in I(v,y)$ are adjacent whenever w' is adjacent to w and v' is adjacent to v. Then $w' \in S$, contrary to the choice of v. Now assume that $w \in I(v,y)$ and $v \in I(w,x)$. By (α_2),

$$d(x,u) \geqslant d(x,v) + d(v,u) - 2 \geqslant 2k-2.$$

As above we deduce that necessarily $d(x,u) = 2k-1$. Then there exist a neighbour $v' \in I(v,x)$ of v, a neighbor $w' \in I(w,u)$ of w, and a vertex p adjacent to w', w, v, and v'. Since $I(w,x) \subseteq I(y,x)$, we obtain $p \in S$ and $d(u,p) < d(u,v)$, contrary to the assumption that $v \in Pr(u,S)$.

Case 2: $w \notin I(x,y)$. Then $v \in I(w,x) \cup I(w,y)$. First, let $v \in I(w,y)$. By Lemma 5, $d(y,u) \geqslant 2k-1$ and the equality holds only if any neighbours $v' \in I(v,y)$ of v and $w' \in I(w,u)$ of w are adjacent. If $d(w,x) = d(v,x)$, by the triangle condition, there exists a common neighbour p of v and w at distance $k-2$ to x. Then either the vertices p, w, v, w', v' induce a house or p and w' are adjacent. In the second case we have $w' \in S$, contrary to $v \in Pr(u,S)$. So, assume that $v \in I(w,x)$. Again, by (α_2),

$$d(x,u) \geqslant d(x,v) + d(v,u) - 2 \geqslant 2k-2.$$

If $d(x,u) = 2k-2$, we immediately get $w' \in S$, which is impossible. Otherwise, if $d(x,u) = 2k-1$, the edge vw belongs to a 3-fan induced by some vertices $w'' \in I(w,u), v'' \in I(v,x)$ and t. As t, w, v, w', v' cannot induce a house, either t and v' or t and w' must be adjacent. In the either case $t \in S$ and $d(t,u) < d(v,u)$, contrary to the choice of v. So, t and w' are adjacent. To avoid a house induced by v'', t, v, w', v', we must have the edge $v''w'$. This implies $w' \in I(x,y)$, i.e., $w' \in S$, contrary to the choice of v.

So, we can suppose that $v \in I(w,x)$ and $d(w,y) = d(v,y)$. By the triangle condition we will find a common neighbour z of v and w one step closer to y. Notice that

$w \in I(z,u)$, otherwise $z \in I(v,u)$ and we are in the conditions of Case 1. Since, in addition, $v \in I(x,w)$, (α_2) implies that

$$d(u,x) \geqslant d(u,w) + d(v,x) - 1 \geqslant 2k - 2,$$

$$d(u,y) \geqslant d(u,w) + d(y,z) - 1 \geqslant 2k - 2,$$

If $d(u,x) = 2k - 2$ or $d(y,u) = 2k - 2$, then any neighbor $w' \in I(w,u)$ of w must be adjacent to any neighbour $v' \in I(v,x)$ or to any neighbour $z' \in I(z,y)$ of z. As a result we will get a house induced by the vertices w', v', v, w, z or by the vertices w', z', z, w, v. Thus, $d(x,u) = d(y,u) = 2k - 1$. By Lemma 5, we can find the vertices $w', w'' \in I(w,u)$ adjacent to w, and the vertices s and t such that s is adjacent to v, w, w' and a neighbour $v' \in I(v,x)$ of v, while t is adjacent to z, w, w'' and a neighbour $z' \in I(z,y)$ of z. By weak modularity of G we can find a common neighbour u^+ of w' and w'' one step closer to u. Note that s, w'' and t, w' cannot be adjacent, otherwise we obtain an induced 5-cycle or a house. Therefore, w' and w'' must be adjacent. The path $w''w'sv$ is induced, otherwise one of our distance requirements is violated. Since $v, w'' \in D(y,k)$ and $s \notin D(y,k)$, we have obtained a contradiction to Lemma 1. □

Corollary 1. $r(G) - 1 \leqslant R \leqslant r(G)$.

Proof. By Lemma 6, we have $2r(G) - 3 \leqslant delta \leqslant 2r(G)$. If $delta$ is odd ($delta = 2k - 1$), then either $delta = 2r(G) - 3$ or $delta = 2r(G) - 1$, thus $r(G) - 1 \leqslant k \leqslant r(G)$. Since $R = k$ in this case, the required inequalities hold. Let now $delta$ is even ($delta = 2k$). Then, either $delta = 2r(G) - 2$ or $delta = 2r(G)$ and again $r(G) - 1 \leqslant k \leqslant r(G)$. Hence, the required inequalities follow from Lemma 8. □

Lemma 10. *There exist vertices $a \in \cap_{p \in S} I(x,p)$ and $b \in \bigcap_{p \in S} I(y,p)$ which are adjacent to all vertices of S. In particular, $d(p', p'') \leqslant 2$ for any vertices $p', p'' \in S$.*

Proof. The set S is m^3-convex and, moreover, $Pr(x,S) = S = Pr(y,S)$. So, we are in position to apply Lemma 4. □

Lemma 11. *Let u, v be vertices of G such that $v \in S$, $u \notin S$ and $d(u,v) > d(u,S)$. Then there exists a vertex $w \in I(u,v) \cap Pr(u,S)$.*

Proof. Pick a vertex $w \in Pr(u,S)$ and assume that $w \notin I(v,u)$. Evidently, this is possible if w and v were nonadjacent. Then $d(v,w) = 2$ by Lemma 10. Since $d(v,u) > d(w,u)$ we deduce that $d(v,u) = d(w,u) + 1$. Hence, we can apply Lemma 3. According to that result we can find two vertices t and z such that t is adjacent to v, w and z, while z is adjacent to w and is one step closer to u. Consider the vertices a and b as described in Lemma 10. Let say $t \neq b$. In order to avoid a house induced by v, t, w, z and b, the vertex b must be adjacent to t or z.

If b and t are adjacent, then necessarily $t \neq a$. Applying the same argument, we conclude that a must be adjacent to t or z. If a and t are adjacent, then $t \in I(x,y)$. Moreover, $t \in S \cap I(u,v)$. Since $d(t,u) = d(w,u)$, the vertex t has the desired property.

So, assume that a and t are nonadjacent, while a and z are adjacent. In this case either a,b,v,w,z induce a house, or z and b must be adjacent. In the second case $z \in I(a,b) \subseteq I(x,y)$. Hence, $z \in S$ and $d(u,z) < d(u,w)$, contrary to the choice of w.

Finally, suppose that b is adjacent to z and is nonadjacent to t. Applying previous arguments, we deduce that a must be adjacent to z or $t = a$. In both cases, $z \in S$ and again, since $d(u,z) < d(u,w)$, we are in contradiction with the choice of w. □

Lemma 12. *If $e(c) > \lfloor delta/2 \rfloor + 1$, then $e(u) > delta$ for any vertex $u \in F(c)$.*

Proof. Let $k = \lfloor (delta + 1)/2 \rfloor$ and recall that $k \leqslant R \leqslant k + 1$. According to Lemmas 8 and 9, $e(c) > R$. By Lemma 11, we can find a vertex $w \in I(c,u) \cap Pr(u,S)$. We distinguish between two cases depending on the value of $d(c,w)$.

Case 1: $d(c,w)=1$. Necessarily $d(u,w)=R$ and $u \in F'$. Since c belongs to a maximum number of metric projections of the vertices of F' on S and $c \notin Pr(u,S)$, we can find a vertex $t \in F'$ such that $c \in Pr(t,S)$ and $w \notin Pr(t,S)$. This implies $d(c,t) = R$ and $c \in I(w,t)$. By (α_2),

$$d(u,t) \geqslant d(t,c) + 1 + d(u,w) - 2 = 2R - 1.$$

We may assume that $delta = 2k - 1$ and $R = k$. Indeed, if $delta = 2k$, then from $R+1=e(c) > k+1$ we would get $R > k$. But if $R=k+1$, then $d(u,t) \geqslant 2k+1 > delta$ and we are done.

So, let $delta = d(u,t) = 2k - 1$ and $R = k$. By Lemma 5, any neighbour $w' \in I(w,u)$ of w is adjacent to any neighbour $c' \in I(c,t)$ of c. Let a and b be the vertices defined in Lemma 10. Each of them must be adjacent to at least one of the vertices c' and w', otherwise we obtain an induced house. From the choice of c and w (recall that $c \in Pr(t,S)$ and $w \in Pr(u,S)$) we deduce that a and b cannot be adjacent with the same vertex. Thus, we may assume that a is adjacent to w' and b is adjacent to c'. As a result, we have constructed a house induced by the vertices a, w', c', c and b. This settles Case 1.

Case 2: $d(w,c) = 2$ for any $w \in I(c,u) \cap Pr(u,S)$. Note that for weak bipolarizable graphs (and hence, for chordal graphs) this case is impossible: vertices c,a,w,b with a neighbour $w' \in I(w,u)$ of w and a neighbour $b' \in I(b,y)$ of b induce either a domino or the graph "A". Consequently, *for a weak bipolarizable graph, if $s \in S^*$ and $e(s) > \lfloor delta/2 \rfloor + 1$, then $e(u) > delta$ for any vertex $u \in F(s)$*, i.e., for these graphs, an arbitrary vertex of S^* can be taken as c, and we do not need step 6 of the algorithm.

For HHD-free graphs, pick now a vertex $w \in I(c,u) \cap Pr(u,S)$ which is adjacent to every vertex $f \in D(c,1)$ such that $d(f, Pr(u,S)) = 1$. The existence of such a vertex w follows from the following fact.

Claim 1. *$D(w,1) \cap D(c,1) \subseteq D(w',1) \cap D(c,1)$ or $D(w',1) \cap D(c,1) \subseteq D(w,1) \cap D(c,1)$ holds for every $w,w' \in Pr(u,S)$.*

Proof. Assume that there exist two vertices f and g in $D(c,1)$ such that f is adjacent to w but not to w' and g is adjacent to w' but not to w. By Lemma 4, there exists a common neighbour u' of w,w' at distance $d(u,S) - 1$ to u. Since $d(c,u') = 3$, in the cycle formed by u', w', g, c, f, and w the only possible chords are ww' and fg. In any case, we obtain an induced house or a hole, a contradiction. □

If $u \in F'$, then the choice of the vertex c implies that there exists a vertex $t \in F'$ such that $c \in Pr(t,S)$ and $w \notin Pr(t,S)$. If $u \in F''$, then the choice of c in the algorithm yields that one can find either a vertex $t \in F'$ such that $c \in Pr(t,S)$ and $w \notin Pr(t,S)$, or a vertex $t \in F''$ such that $c \in Pr(t,S)$ and $d(w, Pr(t,S)) > 1$, or a vertex $t \in F''$ such that $d(w, Pr(t,S)) > 1$ but $d(c, Pr(t,S)) = 1$.

Subcase 2.1: $u,t \in F' \cup F''$, $c \in Pr(t,S)$, $w \notin Pr(t,S)$, and $d(w, Pr(t,S)) > 1$ when $u,t \in F''$.

We have $R - 1 \leqslant d(u,w) \leqslant d(c,t) \leqslant R$, $c \notin Pr(u,S)$, $w \in Pr(u,S)$. Let a and b be the vertices defined in Lemma 10. Since $a,b \in I(c,w)$ and $w \in I(c,u)$, we conclude that $a,b \in I(c,u)$. We assert that $c \in I(a,t) \cap I(b,t)$. Suppose by way of contradiction that $d(a,t) = d(c,t)$. By the triangle condition we can find a common neighbour p of a and c one step closer to t. In order to avoid a house induced by a, c, w, b and p, either p and w or p and b are adjacent. In the either case $d(w,t) = d(c,t)$, contrary to the choice of t. In the second case $p \in I(a,b) \subseteq I(x,y)$, and consequently $p \in S$, again a contradiction. Thus, $c \in I(a,t) \cap I(b,t)$. By the condition (α_2),

$$d(u,t) \geqslant d(u,w) + 2 + d(c,t) - 2 \geqslant d(u,w) + d(c,t).$$

In case of equality, by Lemma 5, the vertex $w \in I(a,u)$ must be adjacent to any neighbour $c' \in I(c,t)$ of c. Then, however, $d(w,t) = d(c,t)$, contrary to the choice of t. Thus,

$$d(u,t) \geqslant d(u,w) + d(c,t) + 1.$$

If $d(u,w) \geqslant k$ and $d(c,t) \geqslant k$, then $d(u,t) \geqslant 2k+1$, and we are done. So, suppose that $d(u,w) \leqslant k-1$. Since $k-1 \leqslant R-1 \leqslant d(u,w) \leqslant k-1$ we conclude that $d(u,w) = k-1$ and $R = k$. Therefore, $e(c) = k+1 = R+1$ and, by Corollary 1, $e(c) \in \{r(G), r(G)+1\}$. Now, if $delta = 2k$, then $k+1 = \lfloor delta/2 \rfloor + 1 < e(c) = d(u,c) = d(u,w) + d(w,c) = k+1$ and a contradiction arises. Hence, we may assume that $d(u,t) = 2k-1 = delta$ and $d(c,t) = k-1$, i.e., both u and t are from F''. In this case, $d(w, Pr(t,S)) > 1$.

Applying Lemma 5 to $d(u,t) = 2k - 1 = d(u,w) + d(c,t) + 1$, we get the vertices $w' \in I(a,u)$ and $c' \in I(c,t)$ such that w', c' lie on a shortest path between u and t, and the vertices w', a, c, c' and some other vertex z induce a 3-fan: see Fig. 2. (Note that $w = w'$ is possible.) To avoid an induced house formed by a, c, b, w, z, the vertex z must be adjacent to w or b. Since w has no neighbours in $Pr(t,S)$, z cannot be adjacent to both of them, otherwise z would be in $Pr(t,S)$ and will be adjacent to w. If z is adjacent to w but not to b, then vertices c', z, c, b, w induce a house. Therefore, $zw \notin E$ and $zb \in E$. Furthermore, from the choice of w we get $w'b \notin E$ (if $w'b \in E$, then $w' \in Pr(u,S)$ but the vertex $z \in D(c,1)$ is adjacent to w' and not to w). Depending on whether w' and w are adjacent or not, we get a house induced by w', a, z, w, b or by w', w, z, b, c.

Subcase 2.2: $u,t \in F''$, $d(c, Pr(t,S)) = 1$ and $d(w, Pr(t,S)) > 1$.

We may assume that $c \in Pr(d,S)$ implies $w \in Pr(q,S)$ for every $q \in F'$, and $d(w, Pr(q,S)) \leqslant 1$ for every $q \in F''$ (see subcase 2.1).

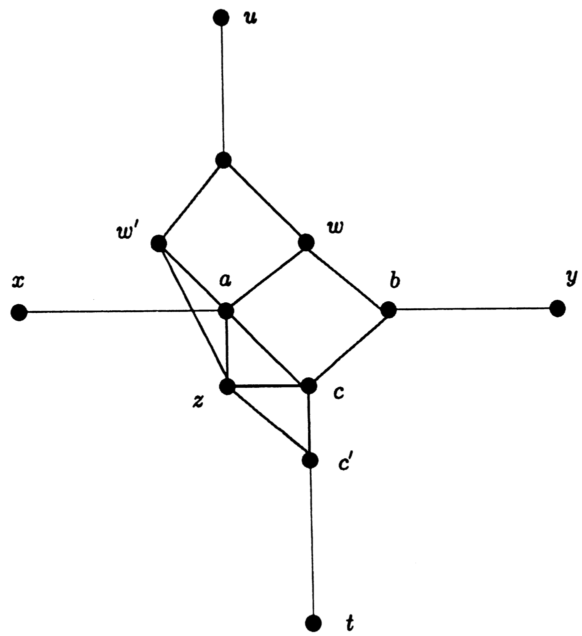

Fig. 2. Illustrations to Lemma 12, subcase 2.1.

Let z be a vertex of $Pr(t,S)$ adjacent to c, and let as before a and b be the vertices defined in Lemma 10. We have $wz \notin E$, $w \in Pr(u,S)$ and $d(c, Pr(u,S)) > 1$.

Claim 2. $d(u,z) = d(u,w) + 2$.

Proof. Since $d(u,c) = d(u,w) + 2$ and $cz \in E$, necessarily $d(u,c) \geqslant d(u,z) \geqslant d(u,w) + 1$. Suppose that $d(u,z) = d(u,w) + 1 = d(u,b)$. By the triangle condition, there exists a vertex w' at distance $d(u,w)$ from u which is adjacent to b and z. If $aw' \in E$, then w' belongs to $Pr(u,S)$. However, the existence of the vertex $z \in D(c,1)$, which is adjacent to w' and not to w, contradicts the choice of w. Thus, $aw' \notin E$. Now we get a house induced by w', w, a, z, b, when $w'w \notin E$, and by $w, w'z, a, c$, otherwise. □

As a consequence of Claim 2, $a, b, w \in I(z,u)$. Similarly to subcase 2.1, we can prove also that $z \in I(a,t) \cap I(b,t)$. So, by (α_2) we infer

$$d(u,t) \geqslant d(u,w) + 2 + d(z,t) - 2 \geqslant d(u,w) + d(z,t) = 2R - 2.$$

Moreover, if equality holds, then the vertex $w \in I(b,u)$ is adjacent to any neighbour $z' \in I(z,t)$ of z, giving a contradiction with $d(w,t) > d(z,t)$. Hence,

$$d(u,t) \geqslant d(u,w) + d(z,t) + 1 = 2R - 1$$

and we may assume again that $d(u,t) = 2R - 1 = 2k - 1 = delta$ (otherwise, we are done).

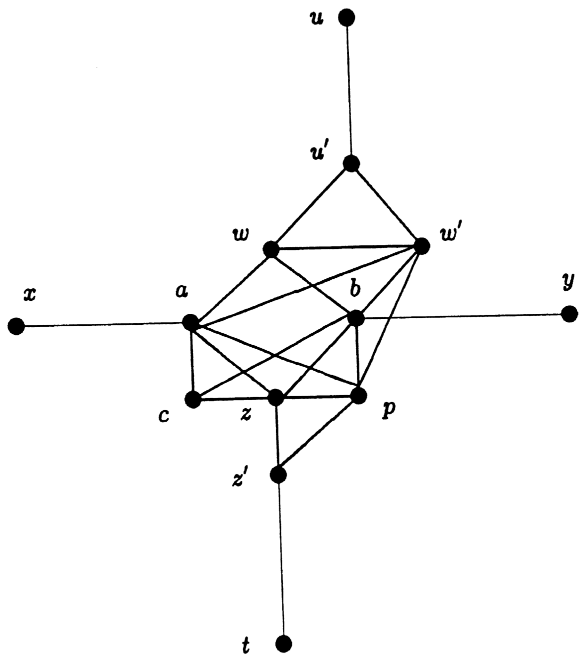

Fig. 3. Illustrations to Lemma 12, subcase 2.2.

Now we apply Lemma 5 to get the vertices $w' \in I(b,u)$ and $z' \in I(z,t)$ such that w', z' lie on a shortest path between u and t, and the vertices w', b, z, z' and some vertex p induce a 3-fan (see Fig. 3). If $pw \in E$, then $pa \in E$ too (otherwise a, w, p, z, z' induce a house). Since now $p \in Pr(t,S)$ is adjacent to w, a contradiction with $d(w, Pr(t,S)) > 1$ arises. So, vertices p and w cannot be adjacent. Applying the quadrangle condition to $w, w' \in I(b,u)$, we will find a vertex u' adjacent to w, w' and at distance $d(u,w) - 1$ from u. To avoid a house induced by p, b, w, u', w', the vertices w and w' must be adjacent. Since G is house- and hole-free, in the cycle formed by w, w', p, z, a, w both chords ap and aw' must be presented. Hence, $w', p \in S$. Note that p cannot be adjacent to c because of the choice of w. Thus, we have constructed an induced subgraph of G shown in Fig. 3.

Consider the vertices c and p from S. We have $u \in F''$, $d(p, Pr(u,S)) = 1$, but $d(c, Pr(u,S)) > 1$. From the choice of c there must be a vertex $q \in F' \cap F''$ such that $c \in Pr(q,S)$ and $p \notin Pr(q,S)$, if $q \in F'$, or $d(c, Pr(q,S)) \leqslant 1$ and $d(p, Pr(q,S)) > 1$, if $q \in F''$.

First assume $c \in Pr(q,S)$. In view of Subcase 2.1, we may suppose that $w \in Pr(q,S)$, if $q \in F'$, and $d(w, Pr(q,S)) \leqslant 1$, if $q \in F''$.

If $w \in Pr(q,S)$, by Lemma 4, there is a vertex c' adjacent to w, c and at distance $d(c,q) - 1$ from q. Since $p \notin Pr(q,S)$, the vertices c' and p are not adjacent. Consequently, the cycle formed by c', w, w', p, z, c may have only chords $c'w'$, $c'z$, and we cannot avoid a forbidden house or a hole.

If $d(w, Pr(q,S)) = 1$, then necessarily $q \in F''$ and hence $d(p, Pr(q,S)) > 1$. Lemma 3 applied to q, w and c will give two vertices s and c' such that $sw, sc, sc', cc' \in E$ and $d(c', q) = d(c, q) - 1$. Since G is house- and hole-free, in the cycle formed by s, w, w', p, z, c all three possible chords sw', sp, sz must exist. From $d(p, Pr(q,S)) > 1$, $sp \in E$, $c \in Pr(q,S)$ and $d(s,q) = d(c,q)$ we conclude that $s \notin S$. Therefore, s cannot be adjacent to both a and b. Let $sa \notin E$. Then, depending on whether c' and a are adjacent, we obtain a house formed by s, w, a, c, c' or by a, w, b, c, c'. Note that $c' \notin S$, therefore c' cannot be adjacent to both a and b.

Finally, assume that $d(c, Pr(q,S)) = 1$ but $d(p, Pr(q,S)) > 1$. Let f be a vertex of $Pr(q,S)$ adjacent to c. We have $af, bf \in E$ and $fp \notin E$. Recall also that $u, q \in F''$, $delta = 2k-1$ and $R = k$. If $fw \in E$ then vertices f, w, w', p, z, c induce a hole or a house (fw' and fz are the only chords of the cycle formed by those vertices). Hence $fw \notin E$. Similarly, f and w' cannot be adjacent.

We claim that $f \in I(a,q)$ and $a \in I(f,u)$. If $f \notin I(a,q)$, then $d(f,q) = d(a,q)$. By the triangle condition, there is a vertex f' which is adjacent to a, f and at distance $d(f,q) - 1$ to q. Since $f \in Pr(q,S)$, $p \notin Pr(q,S)$ and $fp \notin E$, we deduce that the house formed by f', a, p, b, f is induced. Let now $a \notin I(f,u)$, i.e., $d(u,f) \leqslant d(u,a)$. Since $fc \in E$, the vertex f cannot be in $Pr(u,S)$, hence $d(u,a) = d(u,f)$. Again, by the triangle condition, there is a vertex w'' which is adjacent to f, a and at distance $d(u,w)$ from u. From distance requirements we have $w''c \notin E$. If w'' and b are adjacent, then w'' belongs to $Pr(u,S)$ and the existence of $f \in D(c,1)$, which is adjacent to w'' and not to w, contradicts the choice of w. Therefore, $w''b \notin E$ and we obtain a house induced by w'', f, b, w, a, if $w''w \notin E$, or by w'', w, b, c, f, otherwise. Analogously, we can show that $b \in I(f,u)$ and $f \in I(b,d)$.

Now we apply condition (α_2) to $f \in I(a,q)$, $a \in I(f,u)$ and get

$$d(u,q) \geqslant d(u,a) + d(f,q) + 1 - 2 = d(u,w) + d(f,q) = 2k - 2.$$

If $d(u,q) = d(u,w) + d(f,q) = 2k-2$ then, by Lemma 5, the vertex $w \in I(a,u)$ is adjacent to any neighbour $f' \in I(f,q)$ of f, and hence we obtain an induced house on f', w, a, f, c. Consequently, $d(u,d) \geqslant 2k-1$, and we may assume again that $d(u,q) = 2k - 1 = delta$. By Lemma 5, there must be two vertices $w'' \in I(a,u)$ and $f' \in I(f,q)$ such that w'', f' lie on a shortest path between u and q, and the vertices w'', a, f, f' and some vertex s induce a 3-fan. Note that from distance requirements c and w'' are not adjacent. If $sw \in E$, then $sb \in E$ too (otherwise b, w, q, f, f' induce a house). Hence, s belongs to $Pr(d,S)$ and cannot be adjacent to p with $d(p, Pr(q,S)) > 1$. Since the cycle formed by f, s, w, w', p, z, c may have only sw', sz, sc, fz as chords, we cannot avoid a forbidden house or holes. Thus, vertices s and w are not adjacent. Similarly, one can show that s is not adjacent to w'.

Applying the quadrangle condition to $w', w'' \in I(a,u)$, we will find a vertex u'' adjacent to w'', w' and at distance $d(u,w) - 1$ from u. To avoid a house on s, a, w', u'', w'', the vertices w'' and w' must be adjacent. But then vertex b is adjacent to both w'' and s, otherwise w'', w', b, f, q induce a hole or a house. Consequently, w'' belongs to $Pr(u,S)$, and s belongs to $Pr(q,S)$. From $d(p, Pr(q,S)) > 1$, we get $sp \notin E$, and from $z \in D(c,1), zw \notin E, sw'' \in E, sw \notin E$ and the choice of w we infer $zw'', sc \notin E$. Thus, the cycle formed by f, s, w'', w', p, z, c may have only the chords $sz, w''p, fz$, and we cannot avoid a forbidden subgraph. This finishes the proof of the lemma. □

From the proof of this lemma we get the following.

Corollary 2. *Let s be an arbitrary vertex of S^*. Then $e(s) \in \{r(G), r(G)+1\}$ or $e(u) > delta$ for every vertex $u \in F(s)$.*

Lemma 13. *If $e(c) \leqslant \lfloor delta/2 \rfloor + 1$, then $c \in C(G)$.*

Proof. First assume that $delta = 2k - 1$. Then $e(c) \leqslant k$. By Lemma 9 and Corollary 1, we have $e(c) \leqslant R \leqslant r(G)$, i.e., $c \in C(G)$. Now let $delta = 2k$, i.e., $e(c) \leqslant k+1$. By Lemma 6, $r(G) - 1 \leqslant k \leqslant r(G)$ holds. If $k = r(G) - 1$ or $e(c) < k+1$, then $e(c) \leqslant r(G)$ and we are done. So, let $delta = 2r(G)$ and $e(c) = k+1 = r(G)+1$. Then, Lemma 8 gives $R = r(G) = k$. Therefore, all central vertices of G are in S. For each vertex z of $C(G) \subseteq S$, we have $z \in Pr(u, S)$, if $u \in F'$, and $d(z, Pr(t, S)) \leqslant 1$, if $t \in F''$. Consequently, the vertex c with $e(c) = R + 1$ cannot be chosen by the algorithm. □

Analogously, one can show that *if G is a weak bipolarizable graph and s is a vertex from S^* with $e(s) \leqslant \lfloor delta/2 \rfloor + 1$, then $s \in C(G)$.*

3.3. Results

Summarizing we have following results.

Theorem 1. *A central vertex of a HHD-free graph G can be found in time $O(\Delta^{\alpha-1}|V|)$. A vertex s of G with eccentricity $r(G)$ or $r(G)+1$ can be found in linear time $O(|V| + |E|)$.*

Theorem 2. *A central vertex of a weak bipolarizable graph (and hence, of a chordal graph) can be found in linear time.*

Proof. For these graphs we do not need step 6 of the algorithm. Any vertex of S^* can be taken as c. □

Finally, we will show that step 6 of the algorithm can be implemented in linear time if G is a distance-hereditary graph.

Let $u \in V \setminus S$, $v \in S$, and z be an arbitrary vertex of $Pr(u, S)$. We claim that $d(v, Pr(u, S)) \leqslant 1$ if and only if $d(v, \{gate(u), z\}) \leqslant 1$. Suppose there is a vertex $v \in S \setminus Pr(u, S)$ which is adjacent to some vertex w of $Pr(u, S)$ but not to the vertex z. Note that z and w are adjacent to the vertex $gate(u)$ while v is not adjacent. From m^3-convexity of S we infer $wz \in E$. Let, as usual, a and b be the vertices described in Lemma 10. Since $gate(u) \notin S$, we conclude that vertex $gate(u)$ cannot be adjacent to both a and b. Hence, in any case, vertices $gate(u), z, a, v, w$ or vertices $gate(u), z, b, v, w$

induce a 3-fan. Since a distance-hereditary graph cannot contain such an induced subgraph we get the following result.

Theorem 3. *A central vertex of a distance-hereditary graph can be found in linear time.*

We conclude this paper with the following.

Remark. As a consequence of our algorithm we obtain that the interval $I(x,y)$ between any diametral vertices x and y intersects the centre $C(G)$ of a HHD-free graph G. Indeed, according to the algorithm either $S = D(x, \lfloor d(x,y)/2 \rfloor) \cap D(y, d(x,y) - \lfloor d(x,y)/2 \rfloor)$ contains a central vertex of G or we can find a pair of vertices with a larger distance, which is impossible.

Open problem. Find a linear time algorithm for computing a central vertex of a HHD-free graph.

References

[1] H.-J. Bandelt, V.D. Chepoi, A Helly theorem in weakly modular spaces, Discrete Math. 160 (1996) 25–39.

[2] V.D. Chepoi, Some properties of d-convexity in triangulated graphs, Math. Res. (Chişinău) 87 (1986) 164–177 (in Russian).

[3] V.D. Chepoi, Classifying graphs by metric triangles, Metody Diskretnogo Analiza 49 (1989) 75–93 (in Russian).

[4] V.D. Chepoi, F.F. Dragan, Linear-time algorithm for finding a central vertex of a chordal graph, in: Jan van Leeuwen (Ed.), Algorithms—ESA'94, Second Annual European Symposium, Utrecht, The Netherlands, Lecture Notes in Computer Science, Vol. 855, Springer, Berlin, 1994, pp. 159–170.

[5] D. Coppersmith, S. Winograd, Matrix multiplication via arithmetic progression, Proceedings of the 19th ACM Symposium on Theory of Computing, January 1997, New York, pp. 1–6.

[6] F.F. Dragan, Centers of graphs and the Helly property, Ph.D. Thesis, Moldova State University, 1989 (in Russian).

[7] F.F. Dragan, HT-graphs: centers, connected r-domination and Steiner trees, Comput. Sci. J. Moldova 1 (2) (1993) 64–83.

[8] F.F. Dragan, Dominating cliques in distance-hereditary graphs, in: E.M. Schmidt, Sven Skyum (Eds.), Algorithm Theory—SWAT'94, Proceedings of the Fourth Scandinavian Workshop on Algorithm Theory, Aarhus, Denmark, Lecture Notes in Computer Science, Vol. 824, Springer, Berlin, 1994, pp. 370–381.

[9] F.F. Dragan, F. Nicolai, A. Brandstädt, Convexity and HHD-free graphs, SIAM J. Discrete Math. 12 (1999) 119–135.

[10] A.M. Farley, A. Proskurowski, Computation of the center and diameter of outerplanar graphs, Discrete Appl. Math. 2 (1980) 185–191.

[11] G. Handler, Minimax location of a facility in an undirected tree graph, Transportation Sci. 7 (1973) 287–293.

[12] S.M. Hedetniemi, E.J. Cockayne, S.T. Hedetniemi, Linear algorithms for finding the Jordan center and path center of a tree, Transportation Sci. 15 (1981) 98–114.

[13] H. Hempel, D. Kratsch, On claw-free asteroidal triple-free graphs, Proceedings of the 25th International Workshop on Graph-Theoretic Concepts in Computer Science (WG'99), Lecture Notes in Computer Science, Vol. 1665, Springer, Berlin, 1999, pp. 377–390; Discrete Appl. Math. 121 (2002) 155–180.

[14] C.T. Hoàng, N. Khouzam, On brittle graphs, J. Graph Theory 12 (1988) 391–404.

[15] E. Howorka, A characterization of distance-hereditary graphs, Quart. J. Math. Oxford Ser. 2 28 (1977) 417–420.
[16] B. Jamison, S. Olariu, On the semi-perfect elimination, Adv. in Appl. Math. 9 (1988) 364–376.
[17] T. Kloks, private communication.
[18] S. Olariu, Weak bipolarizable graphs, Discrete Math. 74 (1989) 159–171.
[19] S. Olariu, A simple linear-time algorithm for computing the center of an interval graph, Internat. J. Comput. Math. 34 (1990) 121–128.
[20] A. Proskurowski, Centers of 2-trees, Ann. Discrete Math. 9 (1980) 1–5.
[21] B.C. Tansel, R.L. Francis, T. Lowe, Location on networks: a survey.I,II, Manage. Sci. 29 (1983) 482–511.